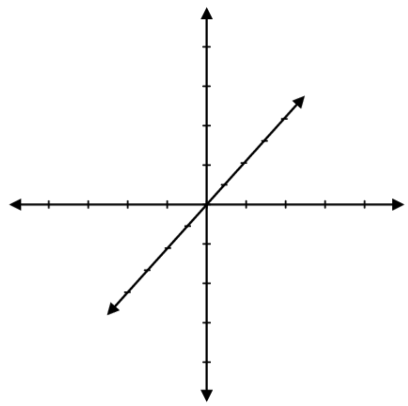
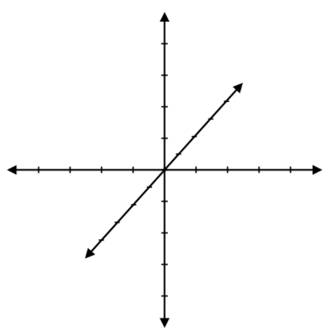
SECTION 13.2: VECTORS IN SPACE

We make our jump to three dimensions by adding a third spatial dimension at a right angle to the two spatial dimensions we are accustomed to. As before, the origin is the point where all the axes intersect. Here, the x-coordinate is how far out (or back) a point is positioned from the origin; the y-coordinate is how far to the right or left the point is from the origin; the z-coordinate is how far above or below a point is from the origin, The positive directions of the axes follow the 'right hand rule' convention.

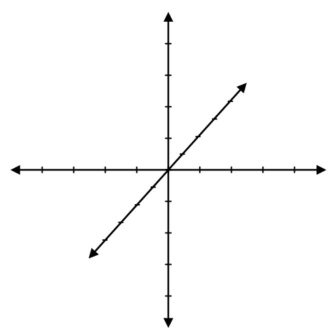
In the figure below, label the axes and plot the points: $P(2, 3, 4)$, $Q(3, -2, -3)$, $R(-3, 2, 2)$, and $O(0, 0, 0)$.



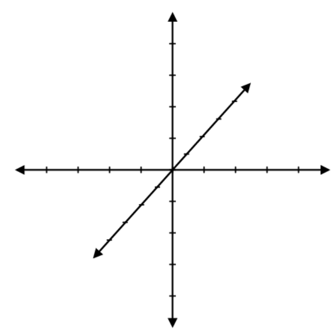
In the figures below, shade the indicated plane:



xy-plane



yz-plane



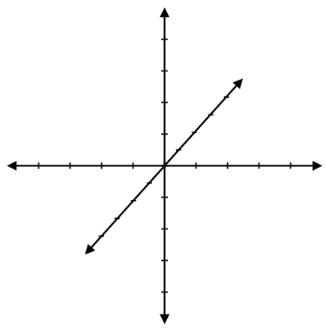
xz-plane

RECALL: What does it mean to graph an equation involving two variables (say x and y)?

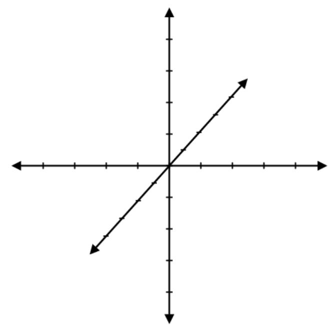
For example, $x = 3$, $y = x^2$, etc.?

GRAPHING EQUATIONS: The graph of an equation involving the variables x, y, and/or z is the set of all points whose coordinates (x, y, z) satisfy the given equation.

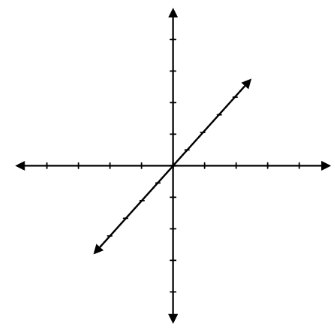
EXAMPLE 1: In the figures below, shade or otherwise describe the graph of the equation.



$z = 2$



$x = 3$



$y = -2$

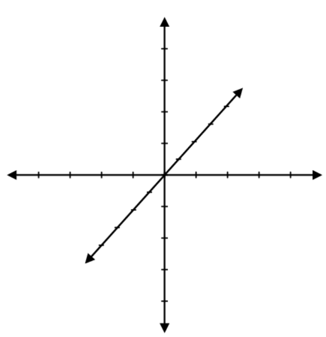
EXAMPLE 2: Determine an equation for the following planes:

1. The xy-plane:

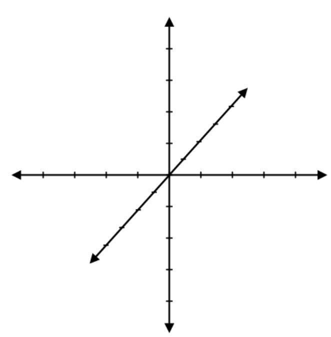
2. The yz-plane:

3. The xz-plane:

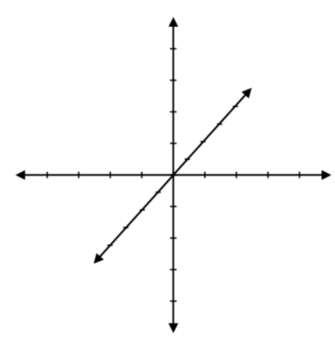
EXAMPLE 3: In the figures below, shade or otherwise describe the graph of the equation.



$x + y = 3$



$(x+1)(z-2) = 0$



$y^2 + z^2 = 9$

DISTANCE AND MIDPOINT FORMULAS:

DISTANCE: We have Pythagoras to thank for the following:

- In the plane: the distance between the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

- In space (thanks to that 90° angle), the distance between the points $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$ is

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

- **NOTE:** On the number line, the distance between x_1 and x_0 is

$$d = |x_1 - x_0| = \sqrt{(x_1 - x_0)^2} = \sqrt{(\Delta x)^2}$$

MIDPOINT:

- In the plane: the midpoint of the line segment connecting $P(x_0, y_0)$ and $Q(x_1, y_1)$ is

$$M = \left(\frac{x_1 + x_0}{2}, \frac{y_1 + y_0}{2}\right) = (\overline{x}, \overline{y}) = (\text{average } x, \text{average } y)$$

- In space: the midpoint of the line segment connecting $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$ is

$$M = \left(\frac{x_1 + x_0}{2}, \frac{y_1 + y_0}{2}, \frac{z_1 + z_0}{2}\right) = (\overline{x}, \overline{y}, \overline{z}) = (\text{average } x, \text{average } y, \text{average } z)$$

- **NOTE:** On the number line, midpoint of the line segment connecting x_1 and x_0 is the average, $\overline{x} = \dfrac{x_1 + x_0}{2}$.

EXAMPLE 4: Let $P(1, 2, 3)$ and $Q(4, -2, 15)$.

1. Find and simplify the distance between P and Q.

Ans: $\sqrt{169} = 13$ units.

2. Find the midpoint of the line segment connecting P and Q.

Ans: $\left(\frac{5}{2}, 0, 9\right)$.

SPHERES:

- **RECALL:** A **circle** centered at (x_0, y_0) with radius r is the set of all points in the plane which are **exactly** r units away from (x_0, y_0). Thanks to the distance formula, the equation describing such a circle is

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

- **RECALL:** A **disk** (**disc**) centered at (x_0, y_0) with radius r is the set of all points in the plane which are **at most** r units away from (x_0, y_0). The equation describing such a disk is

$$(x - x_0)^2 + (y - y_0)^2 \leq r^2$$

- **RECALL:** A **sphere** centered at (x_0, y_0, z_0) with radius r is the set of all points in space which are **exactly** r units away from (x_0, y_0, z_0). Thanks to the distance formula, the equation describing such a sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

- **RECALL:** A **ball** centered at (x_0, y_0, z_0) with radius r is the set of all points in space which are **at most** r units away from (x_0, y_0, z_0). The equation describing such a sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \leq r^2$$

EXAMPLE 5: Work each problem below algebraically and check your answer using a graphing utility.

1. Write the equation of the ball which has $P(1, 2, 3)$ and $Q(4, -2, 15)$ as endpoints of a diameter.

   RECALL: A **diameter** of a sphere is a line segment connecting two antipodal points on the sphere (that is, the line segment connects two points and also contains the center.)

   Ans: $\left(x - \dfrac{5}{2}\right)^2 + y^2 + (z - 9)^2 \leq \dfrac{169}{4}$.

2. Find the center and radius of the sphere: $x^2 + y^2 - 6y + z^2 + 8z = 0$.

   Ans: center: $(0, 3, -4)$, radius: 5.

3. Show that if you 'slice' the sphere $x^2 + y^2 - 6y + z^2 + 8z = 0$ by the plane $x = 3$, the result is a circle.

VECTORS IN SPACE: No surprises! Let $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$. The component form of $\overrightarrow{PQ}$ is

$$\overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle = \langle \Delta x, \Delta y, \Delta z \rangle$$

EXAMPLE 6: Let $P(1, 2, 3)$ and $Q(4, -2, 15)$.

1. Find the component form of $\overrightarrow{PQ}$.

   Ans: $\overrightarrow{PQ} = \langle 3, -4, 12 \rangle$

2. Find the terminal point of $\vec{v}$ if $\vec{v} = \overrightarrow{PQ}$ and the initial point of $\vec{v}$ is $(2, 1, -7)$.

   Ans: $(5, -3, 5)$

3. Find $\|\overrightarrow{PQ}\|$.

   Ans: $\|\overrightarrow{PQ}\| = \sqrt{169} = 13.$

4. Find $\widehat{PQ}$.

   Ans: $\widehat{PQ} = \left\langle \frac{3}{13}, -\frac{4}{13}, \frac{12}{13} \right\rangle$

UNIT VECTORS:

- The x-direction: $\hat{\imath} = \langle 1, 0, 0 \rangle$
- the y-direction: $\hat{\jmath} = \langle 0, 1, 0 \rangle$
- the z-direction: $\hat{k} = \langle 0, 0, 1 \rangle$

EXAMPLE 7: Write the component form of $\vec{v} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$.

Ans: $\langle 3, -2, 1 \rangle$

HOMEWORK: Section 13.2: 9 - 73 every other odd, 85*